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# The grammatical bases

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**Abstract** In this paper, we stumble upon that the normal ordering expansion for $\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^n$ is equivalent to the expansion of $(bD_G)^n$, where $G$ is the context-free grammar defined by $G = \{a \to a,\ b \to 1\}$. Motivated by this fact, we introduce the definition of grammatical basis. We then study several grammatical bases generated by $G = \{a \to 1,\ b \to 1\}$. Using grammatical bases, we give a classification of grammars. In particular, we provide new grammatical descriptions for Ward numbers, Hermite polynomials, Bessel polynomials, Chebyshev polynomials and logarithmic polynomials arising from an integral. We end this paper by giving some applications of grammatical bases. One can see that if two or more polynomials share a grammatical basis, then they share the same coefficients, and it might be helpful for the detection of intrinsic relationship among superficially different structures.

**Keywords** Eulerian numbers, Grammatical bases, Increasing trees, Permutations

## 1 Introduction

The *Weyl algebra* $W$ is the unital algebra generated by two symbols $D$ and $U$ satisfying the commutation relation $DU - UD = I$, where $I$ is the identity which we identify with "1". An example of the Weyl algebra is the algebra of differential operators acting on the ring of polynomials in $x$, generated by $D = \frac{\mathrm{d}}{\mathrm{d}x}$ and $U$ acting as multiplication by $x$. For any $w \in W$, the normal ordering problem is to find the normal ordering coefficients $c_{i,j}$ in the expansion:

$$w = \sum_{i,j} c_{i,j} U^i D^j.$$

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The following expansion has been studied as early as 1823 by Scherk [3, Appendix A]:

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^n=\sum_{k=0}^{n}\left\{ n \atop k \right\}x^k\frac{\mathrm{d}^k}{\mathrm{d}x^k}, \tag{1}$$

where $\left\{ n \atop k \right\}$ is the *Stirling number of the second kind*, i.e., the number of partitions of the set $[n]=\{1,2,\ldots,n\}$ into $k$ blocks. Many generalizations of (1) occur in quantum physics, combinatorics and algebra, see Schork [41] for a survey and see [18–20] for recent progress.

A *context-free grammar* (also known as *Chen's grammar* [8, 15]) $G$ over an alphabet $V$ is defined as a set of substitution rules replacing a letter in $V$ by a formal function over $V$. The formal derivative $D_G$ with respect to $G$ satisfies the derivation rules:

$$D_G(u+v)=D_G(u)+D_G(v),\ D_G(uv)=D_G(u)v+uD_G(v).$$

So the *Leibniz rule* holds:

$$D_G^n(uv)=\sum_{k=0}^{n}\binom{n}{k}D_G^k(u)D_G^{n-k}(v). \tag{2}$$

Recently, context-free grammars have been used extensively in the study of permutations, perfect matchings and increasing trees, see [11, 12, 27, 38] for instances.

In this paper, we always let $D_G$ be the formal derivative associated with the grammar $G$. As an illustration, we now recall the first classical result in this topic.

**Proposition 1.1** ([8]) *If* $G=\{a\to ab,b\to b\}$, *then* $D_G^n(a)=a\sum_{k=0}^n \left\{ n \atop k \right\}b^k$.

The following is a fundamental result of this paper.

**Theorem 1.2** *The expansion* (1) *is equivalent to Proposition 1.1.*

*Proof* Let $G=\{a\to ab,b\to b\}$ and $\widetilde{G}=\{a\to a,\ b\to 1\}$. It is easily proved that $D_G^n(a)=\left(bD_{\widetilde{G}}\right)^n(a)$. Note that (1) can be rewritten as $\left(bD_{\widetilde{G}}\right)^n=\sum_{k=0}^n \left\{ n \atop k \right\}b^kD_{\widetilde{G}}^k$. It readily follows that

$$\left(bD_{\widetilde{G}}\right)^n(a)=\sum_{k=0}^{n}\left\{ n \atop k \right\}b^kD_{\widetilde{G}}^k(a)=a\sum_{k=0}^{n}\left\{ n \atop k \right\}b^k,$$

as desired. This completes the proof.

As suggested in the proof of Theorem 1.2, it is natural to introduce the following definition.

**Definition 1.3** Suppose $u_1(a,b),u_2(a,b),v_1(a,b),v_2(a,b),w_1(a,b)$ and $w_2(a,b)$ are all given functions. Let $G_1=\{a\to u_1(a,b),\ b\to v_1(a,b)\}$ and $G_2=\{a\to u_2(a,b),\ b\to v_2(a,b)\}$. If

$$D_{G_1}^n(w_1(a,b))=\left(w_2(a,b)D_{G_2}\right)^n(w_1(a,b))=f_n(a,b).$$

then we say that $G_2$ is a grammatical basis of $G_1$. We also say that $G_2$ is a grammatical basis of the polynomial $f_n(a,b)$ (or its coefficient sequence).

From the proof of Theorem 1.2, we know that $\widetilde{G}=\{a\to a,\ b\to 1\}$ is a grammatical basis of $G=\{a\to ab,b\to b\}$. The main idea of this paper is stated explicitly in Remark 2.3.

Let $c(n,k)$ be the (signless) type $A$ Stirling number of the first kind, i.e., the number of permutations of the set $[n]$ with $k$ cycles, see [44]. Let $c_B(n,k)$ be the (signless) type $B$ Stirling numbers of the first kind (see [39, Definition 1.4]). They satisfy the recurrence relation:

$$c(n,k)=c(n-1,k-1)+(n-1)c(n-1,k),\ c(0,k)=\delta_{0,k};$$
$$c_B(n,k)=c_B(n-1,k-1)+(2n-1)c_B(n-1,k),\ c_B(0,k)=\delta_{0,k}.$$

It is now well known that

$$x(x+1)(x+2)\cdots(x+n-1)=\sum_{k=0}^{n}c(n,k)x^k;$$
$$(x+1)(x+3)\cdots(x+2n-1)=\sum_{k=0}^{n}c_B(n,k)x^k.$$

As pointed out by Sagan-Swanson [39], $c_B(n,k)$ appears implicitly in a formula of the characteristic polynomial of the intersection lattice of an arbitrary finite complex reflection group.

Another example of Definition 1.3 is given as follows.

**Theorem 1.4** *Let $G=\{a\rightarrow pa, b\rightarrow b\}$, where $p$ is a given parameter. Then $G$ is a common grammatical basis of $\binom{n}{k}$, $c(n,k)$ and $c_B(n,k)$.*

*Proof* By induction, it is routine to verify that for any $n\geqslant 1$, we have

$$D_G^n(ab)=(p+1)^nab,\ (bD_G)^n(ab)=\prod_{k=1}^{n}(p+k)ab^{n+1},\ (b^2D_G)^n(ab)=\prod_{k=1}^{n}(p+(2k-1))ab^{2n+1}.$$

When $p=x, a=b=1$, then obviously

$$D_G^n(ab)|_{p=x,a=b=1}=(x+1)^n=\sum_{k=0}^{n}\binom{n}{k}x^k,$$
$$(bD_G)^n(ab)|_{p=x,a=b=1}=(x+1)(x+2)\cdots(x+n)=\sum_{k=1}^{n}c(n,k)(x+1)^k,$$
$$(b^2D_G)^n(ab)|_{p=x,a=b=1}=(x+1)(x+3)\cdots(x+2n-1)=\sum_{k=0}^{n}c_B(n,k)x^k,$$

and this completes the proof.

This paper is organized as follows. In Section 2, we investigate some grammatical bases generated by the grammatical basis $\{a\rightarrow 1,\ b\rightarrow 1\}$, including $\{a\rightarrow b,\ b\rightarrow b\}$, $\{a\rightarrow b^2,\ b\rightarrow b^2\}$, $\{a\rightarrow ab, b\rightarrow ab\}$ and $\{a\rightarrow ab^2, b\rightarrow ab^2\}$. In Section 3, we first give a classification of several grammars, and we then end this paper by giving some applications of grammatical bases. The advantages of introducing grammatical bases can be summarized as follows:

- In view of the seven Tables given in Section 3, we see that grammars can be systematically discovered;
- As suggested by Corollary 3.3, if two or more polynomials share a grammatical basis, then they can be computed by the same coefficients. It might be helpful for the detection of intrinsic relationship among superficially different structures.

## 2 Grammatical bases generated by the basis $\{a \to 1,\ b \to 1\}$

### 2.1 Notation and preliminaries

The *(type A) Eulerian polynomials* $A_n(x)$ can be defined by the differential expression:

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{1}{1-x} = \sum_{k=0}^{\infty} k^n x^k = \frac{A_n(x)}{(1-x)^{n+1}}.$$

They satisfy the recurrence relation

$$A_n(x) = nxA_{n-1}(x) + x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}A_{n-1}(x),\ A_0(x) = 1. \tag{3}$$

Let $\mathfrak{S}_n$ be the *symmetric group* of all permutations of $[n]$. For $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$, an index $i$ is a *descent* (resp. *excedance*) if $\pi(i) > \pi(i+1)$ (resp. $\pi(i) > i$). Let $\operatorname{des}(\pi)$ and $\operatorname{exc}(\pi)$ be the numbers of descents and excedances of $\pi$, respectively. The *Eulerian polynomials* can also be defined by

$$A_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\operatorname{des}(\pi)+1} = \sum_{\pi\in\mathfrak{S}_n} x^{\operatorname{exc}(\pi)+1} = \sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle x^k,$$

where $\left\langle {n \atop k} \right\rangle$ are known as the *Eulerian numbers* (see [42, A008292]). It is well known that

$$\left\langle {n \atop k} \right\rangle = k\left\langle {n-1 \atop k} \right\rangle + (n-k+1)\left\langle {n-1 \atop k-1} \right\rangle. \tag{4}$$

Using a labeling of circular permutations, Dumont [15] obtained the following result.

**Lemma 2.1** ([15, Section 2.1]) *Let $G = \{a \to ab, b \to ab\}$. Then for $n \geqslant 1$, one has*

$$D_G^n(a) = D_G^n(b) = b^{n+1}A_n\left(\frac{a}{b}\right).$$

Following Carlitz [7], the *second-order Eulerian polynomials* $C_n(x)$ are defined by

$$\sum_{k=0}^{\infty} \left\{ {n+k \atop k} \right\} x^k = \frac{C_n(x)}{(1-x)^{2n+1}},$$

which have been well studied in recent years, see [7, 9, 22, 23, 37]. They satisfy the following recursion (see [7, Eq. (13)]):

$$C_{n+1}(x) = (2n+1)xC_n(x) + x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}C_n(x),\ C_0(x) = 1.$$

In particular, $C_1(x) = x, C_2(x) = x + 2x^2, C_3(x) = x + 8x^2 + 6x^3$. Let $\mathbf{n}_2 = \{1, 1, 2, 2, \ldots, n, n\}$ be a multiset, where each $i$ appears 2 times. We say that a multipermutation $\sigma$ of $\mathbf{n}_2$ is *Stirling permutation* if $\sigma_s > \sigma_i$ as soon as $\sigma_i = \sigma_j$ and $i < s < j$. Denote by $\mathcal{Q}_n$ the set of Stirling permutations of $\mathbf{n}_2$. For $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{Q}_n$, the numbers of ascents, plateaux and descents are respectively defined by

$$\operatorname{asc}(\sigma) = \#\{i \in [2n-1] : \sigma_{i-1} < \sigma_i,\ \sigma_0 = 1\},$$

$$\mathrm{plat}\,(\sigma) = \#\{i \in [2n-1] : \sigma_i = \sigma_{i+1}\},$$

$$\mathrm{des}\,(\sigma) = \#\{i \in \{2,3,\ldots,2n\} : \sigma_i > \sigma_{i+1},\ \sigma_{2n+1} = 0\}.$$

Dumont [14] discovered that the triple statistic (asc , plat , des ) is a symmetric distribution over $\mathcal{Q}_n$, which was independently rediscovered by Bóna [4]. It is now well known that

$$C_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{asc}\,(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{plat}\,(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{des}\,(\sigma)}.$$

Using grammatical labeling of Stirling permutations, Chen-Fu [9] deduced the following result.

**Lemma 2.2** *Let $G = \{a \to ab^2, b \to ab^2\}$. Then for $n \geqslant 1$, one has*

$$D_G^n(a) = D_G^n(b) = b^{2n+1} C_n\left(\frac{a}{b}\right).$$

In this section, we always set

$$G_1 = \{a \to 1,\ b \to 1\},\ G_2 = \{a \to b,\ b \to b\},$$

$$G_3 = \{a \to b^2,\ b \to b^2\},\ G_4 = \{a \to ab, b \to ab\},\ G_5 = \{a \to ab^2, b \to ab^2\}.$$

For any $n \geqslant 1$, it is clear that

$$(abD_{G_1})^n(a) = (aD_{G_2})^n(a) = D_{G_4}^n(a) = b^{n+1} A_n\left(\frac{a}{b}\right),$$

$$(ab^2 D_{G_1})^n(a) = (abD_{G_2})^n(a) = (aD_{G_3})^n(a) = (bD_{G_4})^n(a) = D_{G_5}^n(a) = b^{2n+1} C_n\left(\frac{a}{b}\right).$$

So $G_1$ and $G_2$ are both the common grammatical bases of $A_n(x)$ and $C_n(x)$.

**Remark 2.3** (**Main idea**) Let $f(a,b)$ be a bivariate function. Then $G_1 = \{a \to 1,\ b \to 1\}$ is the grammatical basis of $G = \{a \to f(a,b),\ b \to f(a,b)\}$, since $D_G^n(a) = \left(f(a,b)D_{G_1}\right)^n(a)$. In general, there exists an expansion as follows:

$$(f(a,b)D_{G_1})^n = \sum_{k=1}^{n} F_{n,k}(a,b) D_{G_1}^k.$$

Note that $D_{G_1}(a) = 1$ and $D_{G_1}^2(a) = D_{G_1}(1) = 0$. Then we obtain

$$D_G^n(a) = (f(a,b)D_{G_1})^n(a) = \sum_{k=1}^{n} F_{n,k}(a,b) D_{G_1}^k(a) = F_{n,1}(a,b).$$

Therefore, in order to study $D_G^n(a)$, it is often helpful to investigate $(f(a,b)D_{G_1})^n$, since it may give an interesting refinement of $D_G^n(a)$. Moreover, if two or more polynomials share a grammatical basis, then they can be computed by the same coefficients $F_{n,k}(a,b)$, and so it is promising to further explore the connections among associated combinatorial structures.

### 2.2 On the expansion of $(abD_{G_1})^n$, where $G_1 = \{a \to 1,\ b \to 1\}$

In order to investigate the powers of $abD_{G_1}$, we need to introduce some definitions. The *degree* of a vertex in a tree is referred to the number of its children. We say that $T$ is a *planted full binary increasing plane tree* on $[n]$ if it is a full binary tree with $n+1$ unlabeled leaves and $n$ labeled internal vertices, and satisfying the following conditions (see Figure 1 for examples, where we give every right leaf a weight $b$, and every left leaf a weight $a$):

($i$) Internal vertices are labeled by $1, 2, \ldots, n$. The node 1 is distinguished as the root;

($ii$) Each internal node has exactly two ordered children, which are referred to as a left child and a right child;

($iii$) For each $2 \leqslant i \leqslant n$, the labels of the internal nodes in the unique path from the root to the internal node labelled $i$ form an increasing sequence.

**Definition 2.4** We say that $F$ is a *full binary k-forest* on $[n]$ if it has $k$ connected components, each component is a planted full binary increasing plane tree, the labels of the roots are increasing from left to right and the labels of the $k$-forest form a set partition of $[n]$.


Figure 1: The planted full binary increasing plane trees on [2] encoded by $ab^2D_{G_1}$ and $a^2bD_{G_1}$.

**Theorem 2.5** *Let $G_1 = \{a \to 1, b \to 1\}$. For any $n \geqslant 1$, we have*

$$(abD_{G_1})^n = \sum_{k=1}^{n}\sum_{\ell=k}^{n} f_{n,k,\ell}\, a^{\ell} b^{n+k-\ell} D_{G_1}^k, \tag{5}$$

*where the coefficients $f_{n,k,\ell}$ satisfy the recurrence relation*

$$f_{n+1,k,\ell} = \ell f_{n,k,\ell} + (n+k-\ell+1) f_{n,k,\ell-1} + f_{n,k-1,\ell-1}, \tag{6}$$

*with the initial conditions $f_{1,1,1} = 1$ and $f_{1,k,\ell} = 0$ if $(k,\ell) \neq (1,1)$. The coefficient $f_{n,k,\ell}$ counts full binary k-forests on $[n]$ with $\ell$ left leaves. Moreover, we have*

$$(abD_{G_1})^n = \sum_{k=1}^{n}\sum_{\ell=k}^{\lfloor (n+k)/2 \rfloor} \gamma(n,k,\ell)(ab)^{\ell}(a+b)^{n+k-2\ell} D_{G_1}^k, \tag{7}$$

*where the coefficients $\gamma(n,k,\ell)$ satisfy the recursion*

$$\gamma(n+1,k,\ell) = \ell\gamma(n,k,\ell) + 2(n+k-2\ell+2)\gamma(n,k,\ell-1) + \gamma(n,k-1,\ell-1), \tag{8}$$

*with the initial conditions $\gamma(1,1,1) = 1$ and $\gamma(1,k,\ell) = 0$ for all $(k,\ell) \neq (1,1)$.*

*Proof* (A) The first few $(abD_{G_1})^n$ are given as follows:

$$(abD_{G_1})^2 = (ab^2 + a^2b)D_{G_1} + a^2b^2D_{G_1}^2,$$
$$(abD_{G_1})^3 = (ab^3 + 4a^2b^2 + a^3b)D_{G_1} + (3a^2b^3 + 3a^3b^2)D_{G_1}^2 + a^3b^3D_{G_1}^3,$$
$$(abD_{G_1})^4 = (ab^4 + 11a^2b^3 + 11a^3b^2 + a^4b)D_{G_1} + (7a^2b^4 + 22a^3b^3 + 7a^4b^2)D_{G_1}^2 + (6a^3b^4 + 6a^4b^3)D_{G_1}^3 + a^4b^4D_{G_1}^4.$$

Thus (5) holds for any $n \leqslant 4$. Assume that it holds for $n$. Then we have

$$(abD_{G_1})^{n+1} = abD_{G_1}\left(\sum_{k=1}^{n}\sum_{\ell=k}^{n} f_{n,k,\ell}a^\ell b^{n+k-\ell}D_{G_1}^k\right)$$
$$= \sum_{k=1}^{n}\sum_{\ell=k}^{n} f_{n,k,\ell}\left[\left(\ell a^\ell b^{n+k-\ell+1} + (n+k-\ell)a^{\ell+1}b^{n+k-\ell}\right)D_{G_1}^k + a^{\ell+1}b^{n+k-\ell+1}D_{G_1}^{k+1}\right].$$

Extracting the coefficient of $a^\ell b^{n+k-\ell+1}D_{G_1}^k$ leads to (6), and so (5) holds for $n+1$.

(B) Let $F$ be a full binary $k$-forest. We first give a labeling of $F$ as follows. Label each planted full binary increasing plane tree by $D_{G_1}$, a left leaf by $a$ and a right leaf by $b$. The weight of $F$ is defined to be the product of the labels of all trees in $F$. See Figure 1 for illustrations. Assume that the weight of $F$ is $a^\ell b^{n+k-\ell}D_{G_1}^k$. Let us examine how to generate a forest $F'$ on $[n+1]$ by adding the vertex $n+1$ to $F$. There are three possibilities:

- $c_1$: When the vertex $n+1$ is attached to a leaf with label $a$, then $n+1$ becomes a internal node with two children. The weight of $F'$ is $a^\ell b^{n+k-\ell+1}D_{G_1}^k$;
- $c_2$: When the vertex $n+1$ is attached to a leaf with label $b$, then $n+1$ becomes a internal node with two children. The weight of $F'$ is $a^{\ell+1}b^{n+k-\ell}D_{G_1}^k$;
- $c_3$: If the vertex $n+1$ is added as a new root, then $F'$ becomes a full binary $(k+1)$-forest, the left child of $n+1$ has a label $a$, while the right child of $n+1$ has a label $b$. The weight of $F'$ is given by $a^{\ell+1}b^{n+k-\ell+1}D_{G_1}^{k+1}$.

The above three cases exhaust all the possibilities. Thus $(abD_{G_1})^{n+1}$ equals the sum of the weights of all full binary $k$-forests on $[n+1]$, where $1 \leqslant k \leqslant n+1$.

(C) We now consider a change of the grammar $G_1$. Setting $u = ab$ and $v = a+b$, we get

$$D_{G_1}(u) = D_{G_1}(ab) = v,\ D_{G_1}(v) = D_{G_1}(a+b) = 2.$$

Let $\widetilde{G} = \{u \to v,\ v \to 2\}$. Then we have $(abD_{G_1})^n = \left(uD_{\widetilde{G}}\right)^n$. Note that

$$\left(uD_{\widetilde{G}}\right)^2 = uvD_{\widetilde{G}} + u^2D_{\widetilde{G}}^2,\ \left(uD_{\widetilde{G}}\right)^3 = (uv^2 + 2u^2)D_{\widetilde{G}} + 3u^2vD_{\widetilde{G}}^2 + u^3D_{\widetilde{G}}^3.$$

By induction, we see find that

$$\left(uD_{\widetilde{G}}\right)^n = \sum_{k=1}^{n}\sum_{\ell=k}^{\lfloor (n+k)/2\rfloor} \gamma(n,k,\ell)u^\ell v^{n+k-2\ell}D_{\widetilde{G}}^k,$$

where the coefficients $\gamma(n,k,\ell)$ satisfy the recursion (8). Then upon taking $u = ab$ and $v = a+b$, we get (7). This completes the proof.

Define

$$a_n(x,y,z)=\sum_{k=1}^{n}\sum_{\ell=k}^{n}f_{n,k,\ell}x^{\ell}y^{n+k-\ell}z^k,\ \ a_0(x,y,z)=1.$$

Multiplying both sides of (6) by $x^{\ell}y^{n+k-\ell+1}z^k$ and summing over all $\ell$ and $k$, we obtain

$$a_{n+1}(x,y,z)=x(n+yz)a_n(x,y,z)+x(y-x)\frac{\partial}{\partial x}a_n(x,y,z)+xz\frac{\partial}{\partial z}a_n(x,y,z).$$

In particular, $a_{n+1}(1,1,z)=(n+z)a_n(1,1,z)+z\frac{\mathrm{d}}{\mathrm{d}z}a_n(1,1,z),\ a_0(1,1,z)=1$. Let

$$a_n(1,1,z)=\sum_{k=1}^{n}L(n,k)z^k.$$

It follows that $L(n+1,k)=(n+k)L(n,k)+L(n,k-1)$, from which we notice that $L(n,k)$ is the (signless) *Lah number*. Explicitly, $L(n,k)=\binom{n-1}{k-1}\frac{n!}{k!}$, see [19] for instance.

**Corollary 2.6** *For $n\geqslant 1$, the polynomials $a_n(1,1,z)$ are the Lah polynomials, i.e.,*

$$a_n(1,1,z)=\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{n!}{k!}z^k.$$

A partition of $[n]$ into *lists* is a set partition of $[n]$ for which the elements of each block are linearly ordered. From [42, A008297], we see that $L(n,k)$ counts partitions of $[n]$ into $k$ lists, where a *list* means an ordered subset. Suppose each list is prepended and appended by 0, i.e., we identify a list $\sigma_1\sigma_2\cdots\sigma_i$ with the word $0\sigma_1\sigma_2\cdots\sigma_i0$. An index $p\in\{0,1,2,\ldots,i-1\}$ is an *ascent* of $\sigma_1\sigma_2\cdots\sigma_i$ if $\sigma_p<\sigma_{p+1}$, and $q\in\{1,2,\ldots,i\}$ is a *descent* if $\sigma_p>\sigma_{p+1}$, where we set $\sigma_0=\sigma_{i+1}=0$. Let $F$ be a full binary $k$-forest. Following [43, p. 51], a bijection from full binary $k$-forests to set partitions with $k$ lists can be given as follows: Read the internal vertices of trees (from left to right) of $F$ in symmetric order, i.e., read the labels of the left subtree (in symmetric order, recursively), then the label of the root, and then the labels of the right subtree. Using this correspondence, one can deduce the following result.

**Corollary 2.7** *Let $f_{n,k,\ell}$ be defined by (5). Then $f_{n,k,\ell}$ is the number of set partitions of $[n]$ into $k$ lists with $\ell$ ascents and $n+k-\ell$ descents. In particular, $f_{n,1,\ell}=\left\langle{n \atop \ell}\right\rangle$.*

### 2.3 On the expansion of $(aD_{G_2})^n$, where $G_2=\{a\to b,\ b\to b\}$

We say that $T$ is a *planted binary increasing plane tree* on $[n]$ if it is a binary tree with $n$ unlabeled leaves and $n$ labeled internal vertices, and satisfying the following conditions (where we give every right leaf a weight $b$, and each of the other leaves a weight $a$, see Figure 2):

($i$) Internal vertices are labeled by $1,2,\ldots,n$. The node labelled 1 is distinguished as the root and it has only one child;

($ii$) Excluding the root, each internal node has exactly two ordered children, which are referred to as a left child and a right child;

($iii$) For each $2\leqslant i\leqslant n$, the labels of the internal nodes in the unique path from the root to the internal node labelled $i$ form an increasing sequence.


Figure 2: Planted binary increasing plane trees on [3] encoded by $ab^2D_{G_2}$ and $a^2bD_{G_2}$.

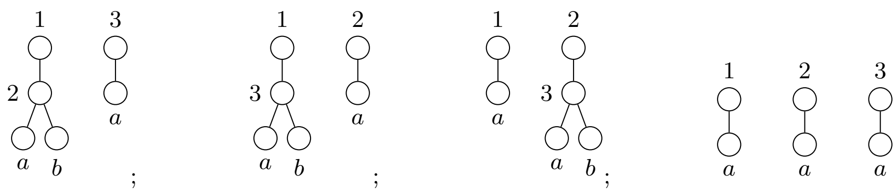


Figure 3: Three 2-forests on [3] encoded by $a^2bD_{G_2}^2$, and the 3-forest on [3] encoded by $a^3D_{G_2}^3$.

**Definition 2.8** We say that $F$ is a *binary $k$-forest* on $[n]$ if it has $k$ connected components, each component is a planted binary increasing plane tree, the labels of the roots are increasing from left to right and the labels of the $k$-forest form a set partition of $[n]$.

**Theorem 2.9** *Let $G_2 = \{a \to b, b \to b\}$. For any $n \geqslant 1$, one has*

$$(aD_{G_2})^n = \sum_{k=1}^{n}\sum_{\ell=k}^{n} h_{n,k,\ell}a^\ell b^{n-\ell}D_{G_2}^k, \tag{9}$$

*where the coefficients $h_{n,k,\ell}$ satisfy the recurrence relation*

$$h_{n+1,k,\ell} = \ell h_{n,k,\ell} + (n-\ell+1)h_{n,k,\ell-1} + h_{n,k-1,\ell-1}, \tag{10}$$

*with $h_{1,1,1} = 1$ and $h_{1,k,\ell} = 0$ if $(k,\ell) \neq (1,1)$. The coefficient $h_{n,k,\ell}$ counts binary $k$-forests on $[n]$ with $n-\ell$ right leaves.*

*Proof* (A) The first few $(aD_{G_2})^n$ are given as follows:

$$(aD_{G_2})^2 = abD_{G_2} + a^2D_{G_2}^2,\ (aD_{G_2})^3 = (ab^2+a^2b)D_{G_2} + 3a^2bD_{G_2}^2 + a^3D_{G_2}^3,$$
$$(aD_{G_2})^4 = (ab^3+4a^2b^2+a^3b)D_{G_2} + (7a^2b^2+4a^3b)D_{G_2}^2 + 6a^3bD_{G_2}^3 + a^4D_{G_2}^4.$$

Thus the expansion (9) holds for any $n \leqslant 4$. Assume that it holds for a given $n$. Since

$$(aD_{G_2})^{n+1} = aD_{G_2}\,(aD_{G_2})^n = aD_{G_2}\left(\sum_{k=1}^{n}\sum_{\ell=k}^{n} h_{n,k,\ell}a^\ell b^{n-\ell}D_{G_2}^k\right),$$

it follows that

$$(aD_{G_2})^{n+1} = \sum_{k=1}^{n}\sum_{\ell=k}^{n} h_{n,k,\ell}\left[\left(\ell a^\ell b^{n-\ell+1} + (n-\ell)a^{\ell+1}b^{n-\ell}\right)D_{G_2}^k + a^{\ell+1}b^{n-\ell}D_{G_2}^{k+1}\right]. \tag{11}$$

Extracting the coefficient of $a^\ell b^{n-\ell+1}D_{G_2}^k$ on both sides yields (10), and so (9) holds for $n+1$.

(B) Let $F$ be a binary $k$-forest. We first give a labeling of $F$ as follows. Label each planted binary increasing plane tree by $D_{G_2}$, a right leaf by $b$, and all the other leaves are labeled by $a$. The weight of $F$ is defined to be the product of the labels of all trees in $F$. See Figures 2 and 3 for illustrations. Suppose the weight of $F$ is $a^\ell b^{n-\ell}D_{G_2}^k$. Let us examine how to generate a forest $F'$ on $[n+1]$ by adding the vertex $n+1$ to $F$. We have the following three possibilities:

- $c_1$: When the vertex $n+1$ is attached to a leaf with label $a$, then $n+1$ becomes a internal node with two children. The weight of $F'$ is $a^\ell b^{n-\ell+1}D_{G_2}^k$;
- $c_2$: When the vertex $n+1$ is attached to a leaf with label $b$, then $n+1$ becomes a internal node with two children. The weight of $F'$ is $a^{\ell+1}b^{n-\ell}D_{G_2}^k$;
- $c_3$: If the vertex $n+1$ is added as a new root, then $F'$ becomes a binary $(k+1)$-forest and the child of $n+1$ has a label $a$. The weight of $F'$ is given by $a^{\ell+1}b^{n-\ell}D_{G_2}^{k+1}$.

As each case corresponds to a term in the right of (11), then $(aD_{G_2})^{n+1}$ equals the sum of the weights of all binary $k$-forests on $[n+1]$, where $1\leqslant k\leqslant n+1$. This finishes the proof.

Comparing (10) with (22), we see that $h_{n+1,1,\ell}=\left\langle {n \atop \ell} \right\rangle$. We define

$$h_n(x,y,z)=\sum_{k=1}^{n}\sum_{\ell=k}^{n}h_{n,k,\ell}x^\ell y^{n-\ell}z^k. \tag{12}$$

Multiplying both sides of (10) by $x^\ell y^{n+1-\ell}z^k$ and summing over all $\ell$ and $k$, we get

$$h_{n+1}(x,y,z)=x(n+z)h_n(x,y,z)+x(y-x)\frac{\partial}{\partial x}h_n(x,y,z),\ h_0(x,y,z)=1. \tag{13}$$

Combining (3) and (13), we find that $h_n(x,1,1)$ equals the Eulerian polynomial $A_n(x)$. Note that the sum of exponents of $x$ and $y$ equals $n$ in a general term $x^\ell y^{n-\ell}z^k$. By induction, it is easy to verify that $yh_n(1,y,1)=A_n(y)$. Using (10), we notice that $h_{n,k,k-1}=0$ and $h_{n+1,k,k}=kh_{n,k,k}+h_{n,k-1,k-1}$. Hence $h_{n,k,k}$ satisfies the same recurrence and initial conditions as $\left\{ {n \atop k} \right\}$, so they agree. In conclusion, we obtain the following result.

**Corollary 2.10** *For $n\geqslant 1$, we have*

$$h_n(1,1,z)=z(z+1)\cdots(z+n-1)=\sum_{k=1}^{n}\begin{bmatrix} n\\ k\end{bmatrix}z^k,\ \sum_{k=1}^{n}h_{n,k,k}z^k=\sum_{k=1}^{n}\begin{Bmatrix} n\\ k\end{Bmatrix}z^k,$$

$$h_n(x,y,1)=y^nA_n\left(\frac{x}{y}\right),\ A_n(x)=h_n(x,1,1)=xh_n(1,x,1).$$

### 2.4 On the expansion of $(aD_G)^n$, where $G=\{a\to b, b\to pb\}$

In [21], Foata-Schützenberger introduced the $q$-Eulerian polynomials

$$A_n(x;q)=\sum_{\pi\in\mathfrak{S}_n}x^{\operatorname{exc}(\pi)}q^{\operatorname{cyc}(\pi)}.$$

They satisfy the recurrence relation (see [6, Proposition 7.2]):

$$A_{n+1}(x;q)=(nx+q)A_n(x;q)+x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}A_n(x;q),\ A_1(x;q)=1. \tag{14}$$

In this subsection, we always write permutation by its standard cycle form, in which each cycle has its smallest element first and the cycles are written in increasing order of their first elements. A *cycle descent* of a permutation is a pair $(a,b)$ where $a$ is the element just before $b$ in its cycle and $a>b$. Let $\mathrm{cdes}\,(\pi)$ be the number of cycle descents of $\pi$. For example, $\mathrm{cdes}\,((1,\mathbf{4},2)(3,5,7)(6,\mathbf{9},8))=2$. In a cycle with $k$ elements ($k\geqslant 2$), the sum of the numbers of excedances and cycle descents equals $k-1$. For $\pi\in\mathfrak{S}_n$, so we have

$$\mathrm{exc}\,(\pi)+\mathrm{cdes}\,(\pi)+\mathrm{cyc}\,(\pi)=n.$$

**Example 2.11** If $\pi=(1,4,2)(3,5,7)(6,9,8)$, then $\mathrm{exc}\,(\pi)=4,\ \mathrm{cdes}\,(\pi)=2$.

We can now present a generalization of Theorem 2.9.

**Theorem 2.12** *Let $G=\{a\to b, b\to pb\}$. For any $n\geqslant 1$, one has*

$$(aD_G)^n|_{D_G=q}=\sum_{\pi\in\mathfrak{S}_n}a^{n-\mathrm{exc}\,(\pi)}b^{\mathrm{exc}\,(\pi)}p^{\mathrm{cdes}\,(\pi)}q^{\mathrm{cyc}\,(\pi)}.$$

*When $p=1$, it reduces to $(aD_G)^n|_{p=1,D_G=q}=h_n(a,b,q)$, where $h_n(x,y,z)$ is defined by (12).*

*Proof* The first few $(aD_G)^n$ are listed as follows:

$$(aD_G)^2=abD_G+a^2D_G^2,\ (aD_G)^3=(ab^2+pa^2b)D_G+3a^2bD_G^2+a^3D_G^3,$$
$$(aD_G)^4=(ab^3+4pa^2b^2+p^2a^3b)D_G+(7a^2b^2+4pa^3b)D_G^2+6a^3bD_G^3+a^4D_G^4.$$

Assume the following expansion holds for a given $n$:

$$(aD_G)^n=\sum_{k=1}^{n}\sum_{\ell=k}^{n}A_{n,k,\ell}(p)a^{\ell}b^{n-\ell}D_G^k. \tag{15}$$

Clearly, $A_{1,1,1}(p)=1$ and $A_{1,k,\ell}(p)=0$ if $(k,\ell)\neq(1,1)$. Since

$$(aD_G)^{n+1}=aD_G\,(aD_G)^n=aD_G\left(\sum_{k=1}^{n}\sum_{\ell=k}^{n}A_{n,k,\ell}(p)a^{\ell}b^{n-\ell}D_G^k\right),$$

it follows that

$$A_{n+1,k,\ell}(p)=\ell A_{n,k,\ell}(p)+(n-\ell+1)pA_{n,k,\ell-1}(p)+A_{n,k-1,\ell-1}(p). \tag{16}$$

which implies that (15) holds for $n+1$. We claim that

$$A_{n,k,\ell}(p)=\sum_{\substack{\pi\in\mathfrak{S}_n\\ \mathrm{exc}\,(\pi)=n-\ell\\ \mathrm{cyc}\,(\pi)=k}}p^{\mathrm{cdes}\,(\pi)}. \tag{17}$$

Given a $\pi'\in\mathfrak{S}_{n+1}$. Suppose $\mathrm{exc}\,(\pi')=n+1-\ell$ and $\mathrm{cyc}\,(\pi')=k$. In order to get $\pi'$ from $\pi\in\mathfrak{S}_n$ by inserting the entry $n+1$, there are three ways:

(*i*) If $\mathrm{exc}\,(\pi) = n-\ell$ and $\mathrm{cyc}\,(\pi) = k$, we can insert $n+1$ right after a drop (i.e., the index $i$ such that $i > \pi(i)$) or a fixed point. Note that there are $\ell$ choices for the position of $n+1$. The first term of the right-hand side of (16) is explained.

(*ii*) If $\mathrm{exc}\,(\pi) = n-\ell+1$ and $\mathrm{cyc}\,(\pi) = k$, we can insert $n+1$ right after an excedance. This means we have $n+1-\ell$ choices for the position of $n+1$. Note that the number of cycle descents will increase by 1. The second term in the right hand side of (16) is explained.

(*iii*) If $\mathrm{exc}\,(\pi) = n-\ell+1$ and $\mathrm{cyc}\,(\pi) = k-1$, we can insert $n+1$ right after $\pi$ as a fixed point. The last term in the right hand side is explained.

This completes the proof of (17).

Combining Theorems 2.9 and 2.12, we immediately get the following result.

**Corollary 2.13** *Let $h_{n,k,\ell}$ de defined by* (9). *Then*

$$h_{n,k,\ell} = \#\{\pi \in \mathfrak{S}_n : \mathrm{cyc}\,(\pi) = k, \quad \mathrm{exc}\,(\pi) = n-\ell\}.$$

### 2.5 On the expansion of $(abD_{G_2})^n$, where $G_2 = \{a \to b,\ b \to b\}$

Consider $G_2 = \{a \to b,\ b \to b\}$. When $n = 2, 3$, we have

$$(abD_{G_2})^2 = (ab^3 + a^2b^2)D_{G_2} + a^2b^2D_{G_2}^2,$$

$$(abD_{G_2})^3 = (ab^5 + 5a^2b^4 + 2a^3b^3)D_{G_2} + 3(a^2b^4 + a^3b^3)D_{G_2}^2 + a^3b^3D_{G_2}^3.$$

Recall that $\mathcal{Q}_n$ is the set of *Stirling permutations* of order $n$. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{Q}_n$. For $1 \leqslant i \leqslant 2n-1$, a value $\sigma_i$ is called a *left-to-right minimum* if $\sigma_i < \sigma_j$ for all $1 \leqslant j < i$ or $i = 1$. Let $\mathrm{lrmin}\,(\sigma)$ be the number of left-to-right minima of $\sigma$. Based on empirical evidence, we can now present the following result.

**Theorem 2.14** *Let $G_2 = \{a \to b,\ b \to b\}$. Then*

$$(abD_{G_2})^n = \sum_{k=1}^{n}\sum_{\ell=k}^{n} g_{n,k,\ell}a^\ell b^{2n-\ell}D_{G_2}^k, \tag{18}$$

*where $g_{n,k,\ell} = \#\{\sigma \in \mathcal{Q}_n : \mathrm{lrmin}\,(\sigma) = k,\ \mathrm{des}\,(\sigma) = \ell\}$.*

*Proof* (A) Note that the expansion (18) holds for $n = 1, 2, 3$. We proceed by induction. Suppose (18) holds for a given $n$. Then we have

$$\begin{aligned}(abD_{G_2})^{n+1} &= abD_{G_2}\left(\sum_{k=1}^{n}\sum_{\ell=k}^{n} g_{n,k,\ell}a^\ell b^{2n-\ell}D_{G_2}^k\right)\\ &= \sum_{k=1}^{n}\sum_{\ell=k}^{n} g_{n,k,\ell}\left[\left(\ell a^\ell b^{2n-\ell+2} + (2n-\ell)a^{\ell+1}b^{2n-\ell+1}\right)D_{G_2}^k + a^{\ell+1}b^{2n-\ell+1}D_{G_2}^{k+1}\right].\end{aligned}$$

Extracting the coefficient of $a^\ell b^{2n-\ell+2}D_{G_2}^k$ on both sides leads to the following recursion:

$$g_{n+1,k,\ell} = \ell g_{n,k,\ell} + (2n-\ell+1)g_{n,k,\ell-1} + g_{n,k-1,\ell-1}, \tag{19}$$

with $g_{1,1,1} = 1$ and $g_{1,k,\ell} = 0$ if $(k,\ell) \neq (1,1)$. Thus the expansion (18) holds for $n+1$.

(B) We claim that $g_{n,k,\ell} = \#\{\sigma \in \mathcal{Q}_n : \mathrm{lrmin}\,(\sigma) = k,\ \mathrm{des}\,(\sigma) = \ell\}$. Given a $\sigma' \in \mathcal{Q}_{n+1}$ with $\mathrm{lrmin}\,(\sigma) = k$ and $\mathrm{des}\,(\sigma) = \ell$. In order to get $\sigma'$ from $\sigma \in \mathcal{Q}_n$ by inserting two copies of $n+1$, there are three ways:

- $(i)$ If $\mathrm{lrmin}\,(\sigma) = k$ and $\mathrm{des}\,(\sigma) = \ell$, we can insert the two copies of $n+1$ right after a descent. Note that there are $\ell$ choices. The term $\ell g_{n,k,\ell}$ is explained.
- $(ii)$ If $\mathrm{lrmin}\,(\sigma) = k-1$ and $\mathrm{des}\,(\sigma) = \ell-1$, we can insert the two copies of $n+1$ just before $\sigma$. The last term in the right hand side of (19) is explained.
- $(iii)$ If $\mathrm{lrmin}\,(\sigma) = k$ and $\mathrm{des}\,(\sigma) = \ell-1$, we can insert the two copies of $n+1$ to one of the remaining positions. This means that we have $2n-(\ell-1)$ choices. The middle term in the right hand side of (19) is explained.

This completes the proof of (19).

### 2.6 On the expansion of $(aD_{G_3})^n$, where $G_3 = \{a \to b^2, b \to b^2\}$.

We say that $T$ is a *planted ternary increasing plane tree* on $[n]$ if it is a ternary tree with $2n-1$ unlabeled leaves and $n$ labeled internal vertices, and satisfying the following conditions (see Figure 4, where we give each leaf a weight):

- $(i)$ Internal vertices are labeled by $1,2,\ldots,n$. The node labelled 1 is distinguished as the root and it has only one child;
- $(ii)$ Excluding the root, each internal node has exactly three ordered children, which are referred to as a left child, a middle child and a right child;
- $(iii)$ For each $2 \leqslant i \leqslant n$, the labels of the internal nodes in the unique path from the root to the internal node labelled $i$ form an increasing sequence.

We say that $F$ is a *ternary $k$-forest* on $[n]$ if it has $k$ connected components, each component is a planted ternary increasing plane tree, the labels of the roots are increasing from left to right and the labels of the $k$-forest form a set partition of $[n]$.

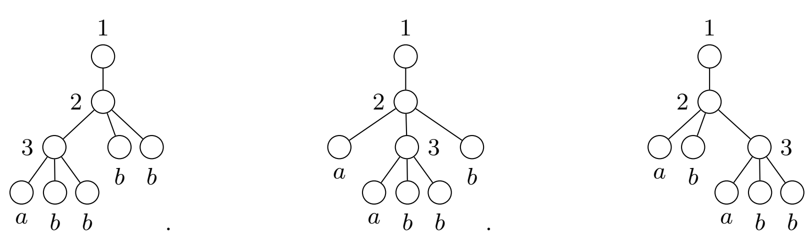


Figure 4: Planted ternary increasing plane trees on [3] encoded by $ab^4D_{G_3}$ and $a^2b^3D_{G_3}$.

Let $F$ be a ternary $k$-forest. We introduce a labeling of $F$ as follows (see Figure 4 for illustrations). Label each planted ternary increasing plane tree by $D_{G_3}$, middle and right leaves are both labeled by $b$, the other leaves are labeled by $a$. If a tree has only one internal vertex and a leaf, then we label the leaf by $a$. Along the same lines as in the proof of Theorem 2.9, it is routine to verify the following.

**Theorem 2.15** *Let $G_3 = \{a \to b^2, b \to b^2\}$. For any $n \geqslant 1$, we have*

$$(aD_{G_3})^n = \sum_{k=1}^{n}\sum_{\ell=k}^{n} C_{n,k,\ell}a^\ell b^{2n-k-\ell}D_{G_3}^k,$$

*where the coefficients $C_{n,k,\ell}$ satisfy the recurrence relation*

$$C_{n+1,k,\ell} = \ell C_{n,k,\ell} + (2n-k-\ell+1)C_{n,k,\ell-1} + C_{n,k-1,\ell-1}, \tag{20}$$

*with $C_{1,1,1} = 1$ and $C_{1,k,\ell} = 0$ if $(k,\ell) \neq (1,1)$. The coefficient $C_{n,k,\ell}$ counts ternary $k$-forests on $[n]$ with $2n-k-\ell$ middle and right leaves. In particular, we have $C_{n+1,1,\ell} = C_{n,\ell}$, where $C_{n,\ell}$ is the second-order Eulerian number, i.e., the coefficient $x^\ell$ in $C_n(x)$.*

Define

$$\widetilde{C}_n(x,y,z) = \sum_{k=1}^{n}\sum_{\ell=k}^{n} C_{n,k,\ell}x^\ell y^{2n-k-\ell}z^k.$$

It follows from (20) that

$$\widetilde{C}_{n+1}(x,y,z) = (xz+2nxy)\widetilde{C}_n(x,y,z) + xy(y-x)\frac{\partial}{\partial x}\widetilde{C}_n(x,y,z) - xyz\frac{\partial}{\partial z}\widetilde{C}_n(x,y,z),$$

with $\widetilde{C}_0(x,y,z) = 1$. When $x = y$, one has

$$\widetilde{C}_{n+1}(x,x,z) = (xz+2nx^2)\widetilde{C}_n(x,x,z) - x^2z\frac{\partial}{\partial z}\widetilde{C}_n(x,x,z), \tag{21}$$

Let $\widetilde{C}(x,x,z;t) = \sum_{n=0}^{\infty}\widetilde{C}_n(x,x,z)\frac{t^n}{n!}$. Then (21) can be written as

$$(1-2x^2t)\frac{\partial}{\partial t}\widetilde{C}(x,x,z;t) = xz\widetilde{C}(x,x,z;t) - x^2z\frac{\partial}{\partial z}\widetilde{C}(x,x,z;t),\ \widetilde{C}(x,x,z;0) = 1.$$

With help of mathematical programming, we find the following result.

**Theorem 2.16** *We have*

$$\widetilde{C}(x,x,z;t) = \mathrm{e}^{xzt\cdot\mathrm{Cat}(x^2t/2)},$$

*where $\mathrm{Cat}(z) = \frac{1-\sqrt{1-4z}}{2z}$ is the generating function for the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$.*

**Corollary 2.17** *For all $n \geqslant 0$, we have*

$$\widetilde{C}_{n+1}(x,x,z) = \sum_{j=0}^{n}\frac{(n+j)!}{2^j(n-j)!j!}x^{n+1+j}z^{n+1-j} = \sum_{j=0}^{n} b(n,j)x^{n+1+j}z^{n+1-j},$$

*where $b(n,j)$ is the Bessel number of first kind [42, A001498].*

*Proof* Using [44, Eq. (2.5.16)], we get

$$\begin{aligned}\widetilde{C}(x,x,z;t) &= \sum_{j\geq 0}\frac{x^j z^j t^j \operatorname{Cat}^j(x^2t/2)}{j!}\\ &= 1+\sum_{j\geq 1}\sum_{i\geq 0}\frac{j}{(i+j)j!2^i}\binom{2i-1+j}{i}x^{2i+j}z^jt^{i+j}\\ &= 1+\sum_{i\geq 0}\sum_{j=0}^{i}\frac{j+1}{(i+1)(j+1)!2^{i-j}}\binom{2i-j}{i-j}x^{2i-j+1}z^{j+1}t^{i+1}.\end{aligned}$$

Hence, for all $n\geqslant 1$, we get

$$\widetilde{C}_n(x,x,z) = n!\sum_{j=0}^{n-1}\frac{j+1}{n(j+1)!2^{n-1-j}}\binom{2n-j-2}{n-1-j}x^{2n-j-1}z^{j+1},$$

which is equivalent to

$$\widetilde{C}_n(x,x,z) = \sum_{j=0}^{n-1}\frac{j!}{2^j}\binom{n-1}{j}\binom{n+j-1}{j}x^{n+j}z^{n-j}.$$

After simplifying, we get the desired explicit formula.

## 3 A classification of context-free grammars and applications

From Table 1 to Table 7, we list some sequences of combinatorial interest and give a classification of the corresponding grammatical descriptions. For each sequence, we give the corresponding entry in [42]. In particular, we provide new grammatical descriptions for Bessel polynomials, Chebyshev polynomials, Hermite polynomials, logarithmic polynomials arising from the integral $\int \mathrm{e}^{\mathrm{e}^{\mathrm{e}^x}}\mathrm{d}x$ [17], the number of partial partitions of $[n+k-1]$ that contain exactly $k$ parts and no singletons [25], the number of derangements over $[n+k]$ with $k$ cycles [29], and Ward numbers [1]. The results in these tables can be proved by induction, and we omit the details for simplicity.

**Table 1** Context-free grammars related to Eulerian numbers

| Grammatical bases | Entry | Description |
|---|---|---|
| $G=\{a\to 1,\ b\to 1\}$ | Eulerian numbers [11, 15], A008292 | $(abD_G)^n(a)$ |
| | $r$-Eulerian numbers [26], A144696, A144697 | $(abD_G)^n(a^r)$ |
| | $r$-order Eulerian numbers [9, 37], A0085177, A219512 | $(ab^rD_G)^n(a)$ |
| $G=\{a\to a,\ b\to c,\ c\to c\}$ | Eulerian numbers, A008292 | $(bD_G)^n(ab)$ |
| | $r$-Eulerian numbers [26], A144696, A144697 | $(bD_G)^n(ab^r)$ |
| | $r$-order Eulerian numbers [9, 37], A0085177, A219512 | $(bc^rD_G)^n(a)$ |
| $G=\{a\to b,\ b\to 1\}$ | Coefficients of André polynomials [15], A094503 | $(aD_G)^n(a)$ |
| | Coefficients of $\frac{1}{2}A_n(2x)$, see (23), A156365 | $(abD_G)^n(a)$ |
| | Coefficients of $x^nA_n^{(2)}\left(\frac{1}{2x}\right)$, see (24), A185410 | $(abD_G)^n(b)$ |

**Table 2** Context-free grammars related to Stirling numbers

| Grammatical bases | Entry | Description |
|---|---|---|
| $G=\{a\rightarrow pa,\ b\rightarrow b\}$ | Stirling numbers of the first kind, $c(n,k)$, see Theorem 1.4, A132393 | $(bD_G)^n(ab)$ |
| | Type $B$ Stirling numbers of the first kind, $c_B(n,k)$, see Theorem 1.4, A039758 | $(b^2D_G)^n(ab)$ |
| $G=\{a\rightarrow a,\ b\rightarrow 1\}$ | Stirling numbers of the second kind [8], A008277 | $(bD_G)^n(a)$ |
| | Lah numbers, $\binom{n}{k}\frac{(n-1)!}{(k-1)!}$, A105278 | $(b^2D_G)^n(a)$ |
| | The independence polynomial of the $n\times n$ rook graph, $k!\binom{n}{k}^2$, A144084 | $(b^2D_G)^n(ab)$ |
| $G=\{a\rightarrow ab,\ b\rightarrow 1\}$ | Bessel polynomials, A100861 | $D_G^n(a)$ |
| | $2^{n-k}{n \brace k}$, A008277 | $(bD_G)^n(a)$ |
| | $2^k{n \brace k}_B$, A154537 | $(bD_G)^n(a^2b)$ |
| $G=\{a\rightarrow ab,\ b\rightarrow -1\}$ | Coefficients of modified Hermite polynomials $2^{-\frac{n}{2}}H_n\left(\frac{x}{\sqrt{2}}\right)$, A096713 | $D_G^n(a)$ |
| | Type $B$ Stirling numbers of the second kind ${n \brace k}_B$, A039755 | $(bD_G)^n\,(ab)$ |
| $G=\{a\rightarrow 2ab,\ b\rightarrow -1\}$ | Coefficients of Hermite polynomials $H_n(x)$, A060821 | $D_G^n(a)$ |
| | $2^k{n \brace k}_B$, A154537 | $(bD_G)^n\,(ab)$ |
| $G=\{a\rightarrow a^2,\ b\rightarrow 1\}$ | $k!{n \brace k}$, A019538 | $(bD_G)^n(a)$ |
| | $(k-1)!{n \brace k}$, A028246 | $(bD_G)^n(ab)$ |
| | Ward numbers [1], A134991 | $(abD_G)^n(a)$ |
| | The number of partial partitions of $[n+k-1]$ that contain exactly $k$ parts and no singletons [25], A124324 | $(abD_G)^n(b)$ |
| | The number of derangements over $[n+k]$ with $k$ cycles [29], A259456 | $(ab^2D_G)^n(a)$ |

**Table 3** Context-free grammars related to Eulerian and Narayana numbers

| Grammatical bases | Entry | Description |
|---|---|---|
| $G=\{a\rightarrow b,\ b\rightarrow a\}$ | Type $B$ Eulerian numbers [31], A060187 | $(abD_G)^n(ab)$ |
| | 1/2-Eulerian numbers [31], $A_n^{(2)}(x)$, A185410 | $(abD_G)^n(a)$ |
| | Coefficients of $x^nA_n^{(2)}(1/x)$ [31], A185411 | $(abD_G)^n(b)$ |
| | Narayana numbers [35], A001263 | $(a^2b^2D_G)^n(a^2)$ |
| | Type $B$ Narayana numbers [35], A008459 | $(a^2b^2D_G)^n(ab)$ |
| | Coefficients of left peak polynomials [30], A008971 | $(aD_G)^n(a)$ |
| | Coefficients of interior peak polynomials [11, 30], A008303 | $(aD_G)^n(b)$ |
| | $\binom{n+1}{2k}$, A034839 | $(a^2D_G)^n(ab)$ |
| | $\binom{n+1}{2k+1}$, A034867 | $(a^2D_G)^n(b^2)$ |
| | $\binom{n+1}{2k-n}$, A119900 | $(a^2D_G)^n(a^2)$ |

**Table 4** Context-free grammars related to derivative polynomials

| Grammatical bases | Entry | Description |
|---|---|---|
| $G=\{a\rightarrow ab,\ b\rightarrow 1+b^2\}$ | Derivative polynomials of secant function [28], A008294 | $D_G^n(a)$ |
| | Derivative polynomials of tangent function [28], A008293 | $D_G^n(b)$ |
| | Chebyshev polynomials of the first kind, A008310 | $(aD_G)^n(ab)$ |
| | Chebyshev polynomials of the second kind, A008312 | $(aD_G)^n(a^2)$ |
| | $f$-vectors of the simplicial complexes dual to the permutohedra of type $A_n$, A019538 | $(bD_G)^n(a^2)$ |
| | $f$-vectors of the simplicial complexes dual to the permutohedra of type $B_n$, A145901 | $(bD_G)^n(ab)$ |
| | One Galton triangle $2^{n-2k}\binom{2k}{k}k!\left\{{n \atop k}\right\}$, A187075 | $(bD_G)^n(a)$ |
| | Another Galton triangle, A186695 | $(bD_G)^n(b)$ |

**Table 5** Context-free grammars related to up-down run polynomials

| Grammatical bases | Entry | Description |
|---|---|---|
| $G=\{a\rightarrow a,\ b\rightarrow c,\ c\rightarrow b\}$ | Up-down run polynomials [32, 45], A186370 | $(bD_G)^n(a)$ |
| | Alternating run polynomials [32, 45], A059427 | $(bD_G)^n(a^2)$ |
| | Flag descent polynomials [31], A101842 | $(bcD_G)^n(ab)$ |
| | Flag ascent-plateau polynomials [36], A256978 | $(bcD_G)^n(a)$ |
| | Type $B$ Eulerian polynomials [5, 31], A060187 | $(bcD_G)^n(bc)$ |
| | Number of atomic set compositions of size $n$ and of length $i$ [2], A109062 | $(abD_G)^n(ab)$ |
| $G=\{a\rightarrow qa,\ b\rightarrow c,\ c\rightarrow b\}$ | $ab\sum_{k=0}^{n}\binom{2n}{2k}q^{2k}+ac\sum_{k=0}^{n-1}\binom{2n}{2k+1}q^{2k+1}$ A034839, A034867 | $D_G^{2n}(ab)$ |
| | $ac\sum_{k=0}^{n}\binom{2n+1}{2k}q^{2k}+ab\sum_{k=0}^{n}\binom{2n+1}{2k+1}q^{2k+1}$ | $D_G^{2n+1}(ab)$ |

**Table 6** Context-free grammars related to Ramanujan polynomials

| | | |
|---|---|---|
| $G=\{a\rightarrow a^2,\ b\rightarrow b\}$ | Ramanujan polynomials [16], A054589 | $(abD_G)^n(ab)$ |
| | Bessel polynomials [24], A001498 | $(aD_G)^n(ab)$ |
| | Coefficients of logarithmic polynomials arising from the integral $\int \mathrm{e}^{\mathrm{e}^{\mathrm{e}^x}}\mathrm{d}x$ [17], $(k-1)!c(n,k)$, A188881 | $(bD_G)^n(ab)$ |
| | $k!c(n,k)$, A225479 | $(bD_G)^n(a)$ |
| $G=\{a\rightarrow a^2,\ b\rightarrow qb\}$ | Permutation coefficients $\frac{n!}{(n-k)!}$, A008279 | $(D_G)^n(ab)$ |
| | Number of $k$-length walks in the Hasse diagram of a Boolean algebra of order $n$, A090802 | $(D_G)^n(ab^2)$ |

**Table 7** Context-free grammars related to factorial numbers

| | | |
|---|---|---|
| $G=\{a\rightarrow a^2,\ b\rightarrow a\}$ | The polynomials $P_n(x)$ defined by $P_1(x)=1$, $P_{n+1}(x)=x(n+\frac{\mathrm{d}}{\mathrm{d}x})P_n(x)$, A078341 | $(bD_G)^n(a)$ |
| | The polynomials $Q_n(x)$ defined by $Q_1(x)=1$, $Q_{n+1}(x)=Q_n(x)+x(n+\frac{\mathrm{d}}{\mathrm{d}x})Q_n(x)$, A055356 | $(bD_G)^n(b)$ |
| $G=\{a\rightarrow a^2,\ b\rightarrow bc,\ c\rightarrow c^2\}$ | Generate the polynomials $n!ab\sum_{k=0}^n a^kc^{n-k}$ | $D_G^n(ab)$ |
| | Double factorial triangle coefficients $\frac{(2n-k)!}{2^{n-k}(n-k)!}$, A193229 | $(bD_G)^n(ab)$ |

In the sequel, we give some applications of grammatical bases.

### 3.1 An application

Following Savage-Viswanathan [40], the $1/k$-*Eulerian polynomials* $A_n^{(k)}(x)$ are defined by

$$\sum_{n=0}^{\infty}A_n^{(k)}(x)\frac{z^n}{n!}=\left(\frac{1-x}{\mathrm{e}^{kz(x-1)}-x}\right)^{\frac{1}{k}}.$$

In particular, $xA_n^{(1)}(x)=A_n(x)$ and the 1/2-Eulerian polynomials $A_n^{(2)}(x)$ are defined by

$$\sum_{n=0}^{\infty}A_n^{(2)}(x)\frac{z^n}{n!}=\sqrt{\frac{1-x}{\mathrm{e}^{2z(x-1)}-x}}=1+z+(1+2x)\frac{z^2}{2}+(1+10x+4x^2)\frac{z^3}{6}+\cdots.$$

They satisfy the recursion

$$A_{n+1}^{(2)}(x)=(1+2nx)A_n^{(2)}(x)+2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}A_n^{(2)}(x), \tag{22}$$

with $A_0^{(2)}(x)=1$. Some of the combinatorial interpretations of $A_n^{(2)}(x)$ are given as follows:

- Ascent polynomial over the inversion sequences $\{(e_1,\ldots,e_n)\in\mathbb{Z}^n: 0\leqslant e_i\leqslant 2(i-1)\}$ (see [40]);
- Enumerative polynomial of perfect matchings of $[2n]$ by the number of blocks with odd larger elements (see [33]);
- Ascent-plateau polynomial of Stirling permutations in $\mathcal{Q}_n$ (see [34]).

Let $\sigma$ be a Stirling permutation in $\mathcal{Q}_n$. The numbers of *ascent-plateaux* and *left ascent-plateaux* of $\sigma$ are respectively defined by

$$\mathrm{ap}\,(\sigma)=\#\{i\in\{2,3,\ldots,2n-1\}:\sigma_{i-1}<\sigma_i=\sigma_{i+1}\},$$

$$\mathrm{lap}\,(\sigma)=\#\{i\in[2n-1]:\sigma_{i-1}<\sigma_i=\sigma_{i+1},\ \sigma_0=0\}.$$

According to [34], we have

$$A_n^{(2)}(x)=\sum_{\sigma\in\mathcal{Q}_n}x^{\mathrm{ap}\,(\sigma)},\ x^nA_n^{(2)}\left(\frac{1}{x}\right)=\sum_{\sigma\in\mathcal{Q}_n}x^{\mathrm{lap}\,(\sigma)}.$$

Let $G=\{a\to b,\ b\to 1\}$. As listed in Table 1, for $n\geqslant 1$, it is easy to check that

$$(abD_G)^n(a)=\sum_{k=1}^{n}2^{k-1}\left\langle {n \atop k}\right\rangle a^kb^{2n+2-2k}=\frac{1}{2}b^{2n+2}A_n\left(\frac{2a}{b^2}\right), \tag{23}$$

$$(abD_G)^n(b)=a^nbA_n^{(2)}\left(\frac{b^2}{2a}\right). \tag{24}$$

Since $(abD_G)^{n+1}(b)=(abD_G)^n(ab)=\sum_{k=0}^{n}\binom{n}{k}(abD_G)^k(a)(abD_G)^{n-k}(b)$, we obtain

$$(abD_G)^{n+1}(b)=a\left[(abD_G)^n(b)\right]+\sum_{k=1}^{n}\binom{n}{k}(abD_G)^k(a)(abD_G)^{n-k}(b).$$

It is well known that Eulerian polynomials are symmetric, i.e., $x^{n+1}A_n\left(\frac{1}{x}\right)=A_n(x)$. Combining this with (23) and (24), after simplifying, we find the following result.

**Theorem 3.1** *For any $n\geqslant 1$, one has*

$$A_{n+1}^{(2)}(x)=A_n^{(2)}(x)+\sum_{k=1}^{n}\binom{n}{k}2^kA_k(x)A_{n-k}^{(2)}(x).$$

*Since $A_n^{(2)}(1)=(2n-1)!!$, it follows that*

$$(2n+1)!!=(2n-1)!!+\sum_{k=1}^{n}\binom{n}{k}2^kk!(2n-2k-1)!!.$$

### 3.2 Another application

Let $\pm[n]=[n]\cup\{-1,-2,\ldots,-n\}$, and let $B_n$ be the hyperoctahedral group of rank $n$. Elements of $B_n$ are signed permutations of $\pm[n]$ with the property that $\sigma(-i)=-\sigma(i)$ for all $i\in[n]$. The *type B Eulerian polynomials* are defined by

$$B_n(x)=\sum_{\sigma\in B_n}x^{\mathrm{des}_B(\sigma)},$$

where $\mathrm{des}_B(\sigma)=\#\{i\in\{0,1,2,\ldots,n-1\}:\ \sigma(i)>\sigma(i+1)\}$ and $\sigma(0)=0$. They satisfy the recursion (see [5, Eq. (11)]):

$$B_n(x)=(1+(2n-1)x)B_{n-1}(x)+2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}B_{n-1}(x),\ B_0(x)=1. \tag{25}$$

Let $B_n(x)=\sum_{k=0}^{n}B(n,k)x^k$. The type $B$ Eulerian number $B(n,k)$ satisfy the recursion

$$B(n,k)=(1+2k)B(n-1,k)+(2n-2k+1)B(n-1,k-1),\ B(0,0)=1. \tag{26}$$

As listed in Table 3, using (22) and (25), we find that if $G=\{a\to b,\ b\to a\}$, then

$$(abD_G)^n(ab)=ab^{2n+1}B_n\left(\frac{a^2}{b^2}\right),$$

$$(abD_G)^n(a)=ab^{2n}A_n^{(2)}\left(\frac{a^2}{b^2}\right),\ (abD_G)^n(b)=ba^{2n}A_n^{(2)}\left(\frac{b^2}{a^2}\right). \tag{27}$$

**Theorem 3.2** *If $G=\{a\rightarrow b,\ b\rightarrow a\}$, then*

$$(abD_G)^n=\sum_{k=1}^{n}\sum_{\ell=0}^{\lfloor(2n-k)/2\rfloor}p_{n,k,\ell}a^{k+2\ell}b^{2n-k-2\ell}D_G^k, \tag{28}$$

*where the coefficients $p_{n,k,\ell}$ satisfy the recurrence relation*

$$p_{n+1,k,\ell}=(k+2\ell)p_{n,k,\ell}+(2n-k-2\ell+2)p_{n,k,\ell-1}+p_{n,k-1,\ell}, \tag{29}$$

*with $p_{1,1,0}=1$ and $p_{1,k,\ell}=0$ if $(k,\ell)\neq(1,0)$. Moreover, $B_n(x)=\sum_{\ell=0}^n p_{n+1,1,\ell}x^\ell$ and*

$$\sum_{k=1}^{n}\sum_{\ell=0}^{\lfloor(2n-k)/2\rfloor}p_{n,k,\ell}x^{k+2\ell}=\sum_{\sigma\in\mathcal{Q}_n}x^{\mathrm{ap}\,(\sigma)+\mathrm{lap}\,(\sigma)}. \tag{30}$$

*Proof* (A) When $n=2,3$, we have $(abD_G)^2=(ab^3+a^3b)D_G+a^2b^2D_G^2$ and

$$(abD_G)^3=(ab^5+6a^3b^3+a^5b)D_G+3(a^2b^4+a^4b^2)D_G^2+a^3b^3D_G^3.$$

Assume the expansion (28) holds for a given $n$, where $n\geqslant 2$. Then we have

$$\begin{aligned}(abD_G)^{n+1}&=abD_G\left(\sum_{k=1}^{n}\sum_{\ell=0}^{\lfloor(2n-k)/2\rfloor}p_{n,k,\ell}a^{k+2\ell}b^{2n-k-2\ell}D_G^k\right)\\&=\sum_k\sum_\ell p_{n,k,\ell}\left((k+2\ell)a^{k+2\ell}b^{2n-k-2\ell+2}+(2n-k-2\ell)a^{k+2\ell+2}b^{2n-k-2\ell}\right)D_G^k+\\&\sum_k\sum_\ell p_{n,k,\ell}a^{k+2\ell+1}b^{2n-k-2\ell+1}D_G^{k+1}.\end{aligned}$$

Extracting the coefficient $a^{k+2\ell}b^{2n-k-2\ell+2}D_G^k$, we arrive at the desired recursion (29), and so the expansion (28) holds for $n+1$.

(B) Let

$$p_n(x,y,z)=\sum_{k=1}^{n}\sum_{\ell=0}^{\lfloor(2n-k)/2\rfloor}p_{n,k,\ell}x^{k+2\ell}y^{2n-k-2\ell}z^k.$$

It follows from (29) that

$$p_{n+1}(x,y,z)=(xyz+2nx^2)p_n(x,y,z)+x(y^2-x^2)\frac{\partial}{\partial x}p_n(x,y,z),\ p_0(x,y,z)=1.$$

When $y=z=1$, we get

$$p_{n+1}(x,1,1)=(x+2nx^2)p_n(x,1,1)+x(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}p_n(x,1,1), \tag{31}$$

with $p_0(x,1,1)=1,p_1(x,1,1)=x,\ p_2(x,1,1)=x+x^2+x^3,\ p_3(x,1,1)=x+3x^2+7x^3+3x^4+x^5$. Combining (31) with [36, Eq. (16)], we arrive at (30). Comparing (29) with (26), we see that $p_{n+1,1,\ell}=B(n,\ell)$. and the proof of the theorem is complete.

Using (27) and (28), after simplifying, it is routine to verify the following result.

**Corollary 3.3** *We have*

$$A_n^{(2)}(x) = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{ap}(\sigma)} = \sum_{k\geqslant 1}\sum_{\ell\geqslant 0} p_{n,2k,\ell}x^{k+\ell} + \sum_{k\geqslant 1}\sum_{\ell\geqslant 0} p_{n,2k-1,\ell}x^{k+\ell-1},$$

$$x^n A_n^{(2)}\left(\frac{1}{x}\right) = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{lap}(\sigma)} = \sum_{k\geqslant 1}\sum_{\ell\geqslant 0} p_{n,2k,\ell}x^{k+\ell} + x\sum_{k\geqslant 1}\sum_{\ell\geqslant 0} p_{n,2k-1,\ell}x^{k+\ell-1},$$

*where* $p_{n,k,\ell} = \#\{\sigma\in\mathcal{Q}_n : \mathrm{ap}(\sigma)+\mathrm{lap}(\sigma) = k+2\ell\}$. *In other words, we get the following decompositions:*

$$A_n^{(2)}(x) = f_1(x)+f_2(x),\ x^n A_n^{(2)}\left(\frac{1}{x}\right) = f_1(x)+xf_2(x),$$

*where* $f_1(x) = \sum_{k\geqslant 1}\sum_{\ell\geqslant 0} p_{n,2k,\ell}x^{k+\ell}$ *and* $f_2(x) = \sum_{k\geqslant 1}\sum_{\ell\geqslant 0} p_{n,2k-1,\ell}x^{k+\ell-1}$ *are both symmetric polynomials. Furthermore,*

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}A_k^{(2)}(x)A_{n-k}^{(2)}\left(\frac{1}{x}\right), \tag{32}$$

*which implies that the type B Eulerian polynomial* $B_n(x)$ *is symmetric, i.e.,* $B_n(x) = x^n B_n\left(\frac{1}{x}\right)$.

**Problem 3.4** It would be interesting to give bijective proofs of Theorem 3.1 and (32).

## 4 Conclusions

In this paper, we consider the decomposition of a formal derivative as a multiplication of a function with another simpler formal derivative. This decomposition can be used to refine the context-free grammar and thus obtain new refinements of the underlying combinatorial objects.

## Conflict of Interest

The authors declare no conflict of interest.

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